November 9th, 2016 **Name (Please Print)** _____

Probability I- Final Exam - Semester I

Instructions: Maximum time is 3 hours. If you are using a result shown in class then please state it precisely.

1. An urn has four balls labeled $1, 2, 3$, and 4. A first ball is drawn and its number is denoted by X. A second ball is then drawn from the three remaining balls in the urn and its number is denoted by Y.
   (a) (7 points) Find the joint distribution of X and Y.
   (b) (3 points) Are X and Y independent random variables ?
   (c) (10 points) Find the conditional expectation of $Y \mid X = 1$.

2. A random variable X is uniformly distributed over the unit interval $(0, 1)$.
   (a) (5 points) Find the distribution of $Y = -\ln X$, ln denoting the natural logarithm.
   (b) (15 points) Assume that for $0 < a < 1$, there exists $c_1, c_2 > 0$ such that
   $$c_1 a < \ln(1 + a) < c_2 a.$$
   Suppose that $X_1, \cdots, X_n$ are i.i.d uniform $(0, 1)$ random variables. Find the limiting behaviour of $G_n = \left(X_1 \times \cdots \times X_n\right)^{1/n}$ as n tends to ∞.

3. Let X be a discrete random variable.
   (a) (5 points) State Chebyshev's inequality.
   (b) (15 points) Let $\theta > 0$. Using Chebyshevs inequality find
   $$\lim_{n\to\infty} \sum_{j \le n\theta, j \in \mathbb{N}\cup\{0\}} \frac{e^{-n} n^j}{j!}.$$

4. Let X be a Normal random variable with mean μ and non-zero variance σ^2
   (a) (10 points) Let $Y = \frac{X-\mu}{\sigma}$. Find the probability density function of Y.
   (b) (10 points)The heights of women are normally distributed with a mean of 65 inches and a standard deviation of 2.5 inches. Sheela is 65.45 inches tall. Using the normal tables (backside), find the percentage of women that are taller than Sheela.

5. Let $Y \sim$ Exponential(λ).
   (a) (10 points) Calculate the moment generating function $M_Y(t)$.
   (b) (10 points) Use (a) to calculate $E[Y^3]$ and $E[Y^4]$, the third and fourth moments of an exponential distriubtion.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2882	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3079	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3290	0.3315	0.3340	0.3365	0.3389
1.0	0.3414	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3622
1.1	0.3643	0.3665	0.3687	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4083	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4193	0.4207	0.4222	0.4237	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4358	0.4370	0.4382	0.4394	0.4406	0.4418	0.4430	0.4441
1.6	0.4452	0.4463	0.4474	0.4485	0.4495	0.4505	0.4516	0.4526	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4600	0.4608	0.4617	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4679	0.4686	0.4693	0.4700	0.4706
1.9	0.4713	0.4720	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4762	0.4767
2.0	0.4773	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4813	0.4817
2.1	0.4822	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4858
2.2	0.4861	0.4865	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4914	0.4916
2.4	0.4918	0.4920	0.4923	0.4925	0.4927	0.4929	0.4931	0.4933	0.4934	0.4936
2.5	0.4938	0.4940	0.4942	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4954	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4966	0.4967	0.4968	0.4969	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4975	0.4975	0.4976	0.4977	0.4978	0.4978	0.4979	0.4980	0.4980	0.4981
2.9	0.4982	0.4982	0.4983	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4991	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4993	0.4993	0.4993
3.2	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995	0.4995
3.3	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998	0.4998	0.4998

Table 1: Normal tables evaluating : $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{x^2}{2}} dx$